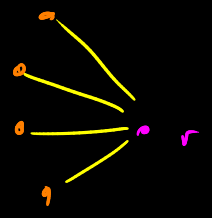
Last Time: Bipartite graphs

Prop: Let G be a graph. The following are equivalent:

1. G is bipartite
2. Every closed walk in G has even length
3. Every cycle in G has even length
4. G is a subgraph of $K_{L,R}$ for some L, R.

Comment on my sketch last time:

Gave an algorithm for deciding if G is bipartite:

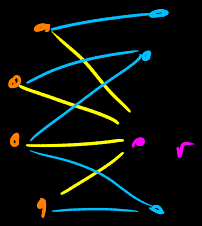


Pick a "root vertex" in each component.

Let $R_0 = \{v \in V(G) : v \text{ is a chosen root}\}$.

$\hookrightarrow L_1 = \bigcup_{v \in R_0} N_G(v)$ (ask: does $L_1 \cap R_0 = \emptyset$?)

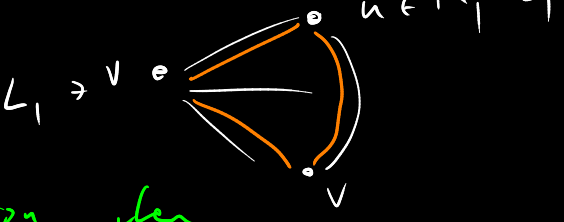
$R_1 = \bigcup_{v \in L_1} N_G(v)$ (ask: does $L_1 \cap R_1 = \emptyset$?)



$\vdots$

$L_i := \bigcup_{v \in R_{i-1}} N_G(v)$ $\hookrightarrow$ if no:

$R_i := \bigcup_{v \in L_i} N_G(v)$



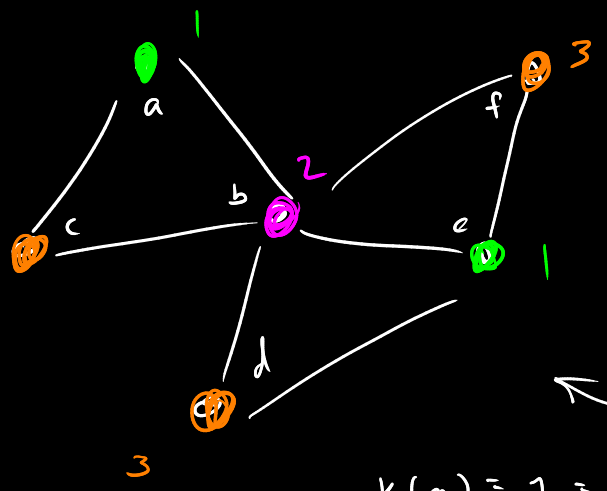
Point: This algorithm constructs a bipartition when possible or produces an odd cycle (verifying no bipartition exists).

Graph Coloring

Defn: Let G be a graph. A (proper) k-coloring of G is a function $c : V(G) \to [k]$ satisfying for all $uv \in E(G)$ $c(u) \neq c(v)$.

(Remark: $[k]$ is the "color set" can be replaced by any set of k elements).

Ex:



"=" $c(v) = 1$

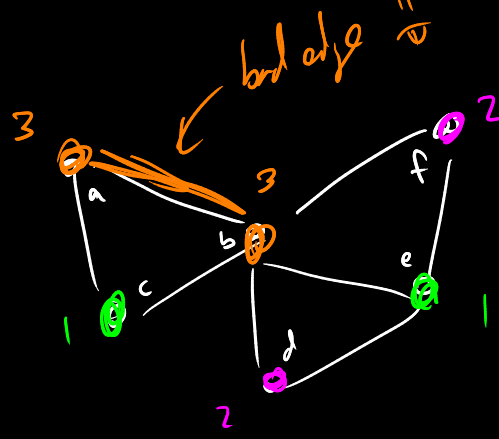
"=" $c(v) = 2$

"=" $c(v) = 3$

← proper coloring of G ☺

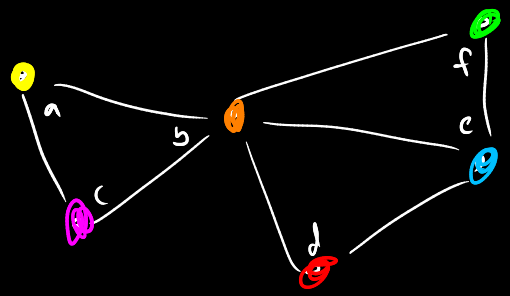
$k(a) = 1 = k(e)$, $k(b) = 2$, $k(c) = k(d) = k(f) = 3$

↑ k is a proper 3-coloring.



← NOT a proper 3-coloring ☹

$k(c) = k(e) = 1$, $k(d) = k(f) = 2$, $k(a) = k(b) = 3$



← proper 6-coloring of G ☺

Prop: Let G be a graph w/ n vertices. Then G has an n-coloring.

pf: $V(G) \xrightarrow{id} V(G)$ is an n-coloring of G. ∎

Prop: Let $H \leq G$. Every proper k-coloring of G yields a proper k-coloring of H.

pf: Let $c : V(G) \to [k]$ be a proper coloring of G, we have $V(H) \subseteq V(G)$ because H is a subgraph of G. Define $\bar{c} : V(H) \to [k]$ via $\bar{c}(v) = c(v)$. Suppose $uv \in E(H)$. Because $H \leq G$, we see $uv \in E(G)$. Thus $\bar{c}(u) = c(u) \neq c(v) = \bar{c}(v)$. Hence $\bar{c}$ is a proper coloring of H as desired. ∎

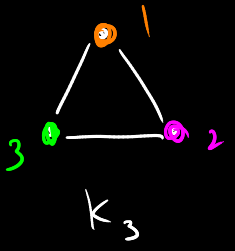
Q: Given a graph G, how many colors are necessary for coloring?

Defn: The chromatic number of graph G is the minimum number of colors in a proper coloring of G. More precisely:

$$\chi(G) := \min\{k \in \mathbb{N} : \exists \text{ proper } k\text{-coloring of } G\}.$$

How can we compute the chromatic number?

Ex: Compute the chromatic number of...

① 

K_3

$\chi(K_3) \leq 3$

To see $\chi(K_3) \geq 3$, assume to the contrary K_3 has a 2-coloring $c: V(K_3) \to [2]$.

Constructing a partition: $V(K_3) = c^{-1}\{1\} \cup c^{-1}\{2\}$. We have 2 subsets of $V(K_3)$ containing a total of $3 = 2+1$ elements... By the pigeonhole principle, one of $c^{-1}\{1\}$ or $c^{-1}\{2\}$ has at least two elements $u \neq v$. But $uv \in E(K_3)$ and $c(u) = c(v)$ by construction so c was not a proper coloring of K_3!

Remark: To prove $\chi(G) = k$, we usually go through 2 steps:

① Produce a proper k-coloring of G (implies $\chi(G) \leq k$).

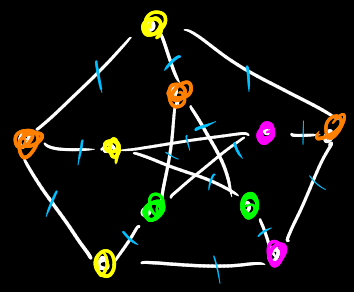
② Prove there is no proper $(k-1)$-coloring of G (implies $\chi(G) \geq k$).

BOTH STEPS ARE NECESSARY HERE!

NB: In general, $\chi(K_n) = n$. We already showed $\chi(K_n) \leq n$ for all n (because every graph w/ n vertices has a proper n-coloring).

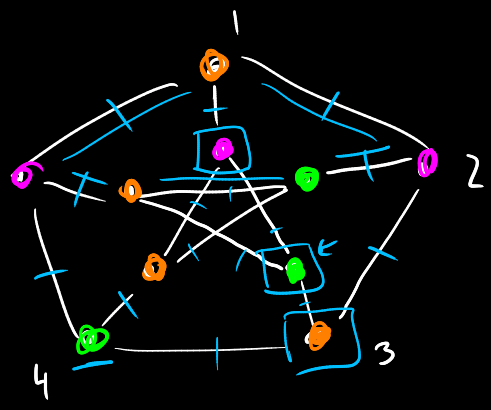
To see $\chi(K_n) \geq n$, assume to the contrary there is a coloring $c: V(K_n) \to [n-1]$. Define $P_i := c^{-1}\{i\}$ for all $1 \leq i \leq n-1$. But $\#V(K_n) = n$ and $\#\{P_i : 1 \leq i \leq n-1\} = n-1$, so the pigeonhole principle yields $\#P_i = \#c^{-1}\{i\} \geq 2$ for some i. Finish...

Ex: We color the Petersen Graph.

$P =$ 

Hence $\chi(P) \neq 4$.

$\chi(P) \leq 4$.



$\chi(P) \leq 3$ by the coloring at the left.

To see $\chi(P) \geq 3$, note there is a $C_5 \leq P$ (on the "outer ring"). Hence $\chi(C_5) \leq \chi(P)$.

Claim: $\chi(C_5) = 3$. (WHY?)

Prop: A graph G is bipartite if and only if $\chi(G) \leq 2$.

pf: If G is bipartite, then $G \leq K_{L,R}$ for some L, R by a previous proposition. On the other hand,

$c: L \cup R \to [2]$ via $c(v) = \begin{cases} 1 & \text{if } v \in L \\ 2 & \text{if } v \in R \end{cases}$

is a proper coloring of $K_{L,R}$ (because $uv \in E(K_{L,R})$ implies — up to relabeling u, v — $u \in L$ and $v \in R$, so $c(u) = 1 \neq 2 = c(v)$). Hence $\chi(G) \leq \chi(K_{L,R}) \leq 2$.

If $\chi(G) \leq 2$, then there is a 2-coloring $c: V(G) \to [2]$. Let $L = c^{-1}\{1\}$ and $R = c^{-1}\{2\}$. For $uv \in E(G)$, $c(u) \neq c(v)$, so u and v are in different parts L, R.

Hence G is bipartite on bipartition $L \cup R$. ☐

Returning to $C_5 \leq P$: C_5 is not bipartite, so $\chi(C_5) \neq 2$, so $\chi(C_5) \geq 3$. So we've shown:

$$3 \leq \chi(C_5) \leq \chi(P) \leq 3$$

↑ not bipartite, hence not 2-colorable.

↑ from subgraph

← explicit coloring

Hence $\chi(P) = 3$ ☺

Ex:

① $\chi(K_n) = n$ for all $n \geq 1$.

② $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$

③ $\chi(P_n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n \geq 1 \end{cases}$

□